Analysis of Spherical Shells of Variable Wall Thickness

BY

MERHYLE F. SPOTTS

Reprinted from the JOURNAL OF APPLIED MECHANICS for Sept. 1939, Trans. A. S. M. E., Vol. 61, p. A-97.

R. UNIV
BIBLIOTHEEK
LEIDEN

R. UNIV.
BIBLIOTHEEK
LEIDEN

Analysis of Spherical Shells of Variable Wall Thickness[1]

By MERHYLE F. SPOTTS,[2] BALTIMORE, MD.

This article presents a solution to the problem of finding the forces and moments which occur in a spherical shell which is axisymmetrically loaded, when the variation in wall thickness is taken as a quadratic function of the coordinate of latitude.

The so-called Love-Meissner differential equations for the case of nonuniform wall thickness are derived herein. By appropriate substitutions these are reduced to one linear equation of the fourth order having constant coefficients. The solution to this equation when taken in the homogeneous form is first given. This homogeneous solution will be the general solution for all problems where the shell surface is free from external force, and the only loads on the shell consist of forces and moments applied at the boundaries.

When, however, the shell surface itself is loaded, a particular integral, which will depend upon the type of loading, must also be obtained for the nonhomogeneous equations.

The solution contained herein is illustrated by a numerical example for a shell of one boundary when acted upon by dead-load forces, and the results are plotted.

Historical

THE MODERN theory for shells having a wall thickness sufficiently great to resist bending may be said to date from Love (1)[3] who in 1892 gave the equilibrium equations for an element of the surface.

Differential equations for a spherical shell of constant wall thickness with axisymmetrical boundary loads were obtained by Reissner (2). These were solved by Blumenthal (3) by an asymptotic integration. Meissner (4) also obtained a solution in the form of a hypergeometric series.

Later Meissner (5) pointed out that the solution obtained for cylindrical shells could be applied to sufficiently thin spherical shells when the angle of opening is large. This is satisfactory in the region near the boundary since the stresses produced by bending are rapidly damped out. Geckeler (6) arrived at this same conclusion by taking the differential equations for a spherical shell and neglecting the terms of less importance.

In regard to the problem of shells of rotation of variable wall thickness, a solution has been presented by Meissner (7) for different variations in thickness and shape of meridian curve. An approximate solution for spherical shells of variable thickness has also been given by Geckeler (6).

Notation

a	=	radius of sphere
d	=	thickness of shell at pole or apex
D	=	$E\delta^3/12(1-\mu^2)$ modulus of flexural rigidity
E	=	Young's modulus
M_1	=	unit bending moment with respect to φ
M_2	=	unit bending moment with respect to θ
N_1	=	unit meridional normal force
N_2	=	unit hoop force
Q	=	unit meridional shearing force
r_0	=	radius of latitude circle
v	=	meridional displacement
w	=	radial displacement
Y	=	unit external load parallel to meridian
Z	=	unit external load normal to surface
γ	=	weight per unit volume of shell
δ	=	thickness of shell
ϵ_1	=	elongation along meridian
ϵ_2	=	elongation along latitude circle
θ	=	coordinate of longitude
κ_1	=	change of curvature with respect to φ
κ_2	=	change of curvature with respect to θ
μ	=	Poisson's ratio
φ	=	coordinate, complement of latitude
χ	=	rotation of meridional tangent

Solution for the Case When the Shell Surface Is Free From External Forces

The usual elastic assumptions are made, namely, that the material is homogeneous, isotropic, and obeys Hooke's law, and that straight lines initially normal to the middle surface remain straight and normal to that surface after bending.

Fig. 1 shows an element cut from a sphere having a vertical axis, and a radius a. The coordinate φ of the element is measured from the axis in a downward direction, and the coordinate θ is measured from any fixed axial plane.

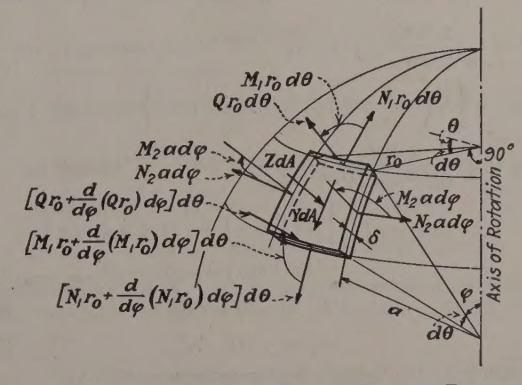

Fig. 1 Element of Spherical Surface With the Forces and Moments Which Produce Equilibrium

---

[1] Taken from a dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the University of Michigan.

[2] Instructor, Department of Mechanical Engineering, Johns Hopkins University.

[3] Numbers in parentheses refer to the Bibliography.

For presentation at the National Meeting of the Applied Mechanics Division of The American Society of Mechanical Engineers, to be held in New York, N. Y., June 14–15, 1939.

Discussion of this paper should be addressed to the Secretary, A.S.M.E., 29 West 39th Street, New York, N. Y., and will be accepted until August 10, 1939, for publication at a later date.

Note: Statements and opinions advanced in papers are to be understood as individual expressions of their authors, and not those of the Society.

The top and bottom edges of the element are formed by two conical surfaces inclined to the axis by the angles φ and $\varphi + d\varphi$. The sides are formed by two meridional sections an angle $d\theta$ apart.

The quantities N_1, N_2, and Q represent force per unit length of edge; M_1 and M_2 represent moment per unit length; and Y and Z represent force per unit area. Positive directions are as shown in Fig. 1. The element, together with all forces, moments, and external loads, is symmetrical with respect to the axis. Thus no other forces or moments than those shown in Fig. 1 need be considered. All forces and moments are thus functions of the coordinate φ only.

As shown in the Appendix it is possible to write three equations of equilibrium for the element together with four more equations expressing relationships between the deformations. These seven equations make it possible to solve for the seven unknowns N_1, N_2, Q, M_1, M_2, v, and w, where v is the displacement of the element along a meridian, positive downward, and w is the displacement along the radius, positive outward.

If the variation in wall thickness is now taken as

$$\delta = d(1 + \alpha\varphi)^2 \quad \ldots\ldots [1]$$

where d is the thickness at the pole, and α is an arbitrarily chosen constant, and the same approximations are made as in the case of constant thickness (10), these equations may be reduced to one linear equation of the fourth order having constant coefficients.

The values of the unknowns,[4] when the equation is taken in the homogeneous form, are found to be[5,6]

$$^{\circ}Q = [d^{1/2}/(\sin\varphi)^{1/2}]\{C_1(1+\alpha\varphi)^{(3/2)+k}\cos[k\log(1+\alpha\varphi) - \gamma_1] + C_2(1+\alpha\varphi)^{(3/2)-k}\cos[k\log(1+\alpha\varphi) - \gamma_2]\} \quad \ldots\ldots [2]$$

$$^{\circ}Q' = \frac{C_1 d^{1/2}}{(\sin\varphi)^{1/2}}(1+\alpha\varphi)^{(1/2)+k}\left\{-\alpha k\sin[k\log(1+\alpha\varphi) - \gamma_1] + \left[\alpha\left(\frac{3}{2}+k\right) - \frac{1}{2}(1+\alpha\varphi)\cot\varphi\right]\cos[k\log(1+\alpha\varphi) - \gamma_1]\right\} + \frac{C_2 d^{1/2}}{(\sin\varphi)^{1/2}}(1+\alpha\varphi)^{(1/2)-k}\left\{-\alpha k\sin[k\log(1+\alpha\varphi) - \gamma_2] + \left[\alpha\left(\frac{3}{2}-k\right) - \frac{1}{2}(1+\alpha\varphi)\cot\varphi\right]\cos[k\log(1+\alpha\varphi) - \gamma_2]\right\} \quad \ldots\ldots [3]$$

$$^{\circ}\chi = \frac{2\alpha^2 k^2}{Ed^{1/2}(\sin\varphi)^{1/2}}\{C_1(1+\alpha\varphi)^{-(5/2)+k}\sin[k\log(1+\alpha\varphi) - \gamma_1] - C_2(1+\alpha\varphi)^{-(5/2)-k}\sin[k\log(1+\alpha\varphi) - \gamma_2]\} \quad \ldots\ldots [4]$$

$$^{\circ}\chi' = \frac{2\alpha^2 k^2 C_1}{Ed^{1/2}(\sin\varphi)^{1/2}}(1+\alpha\varphi)^{-(7/2)+k}\{\alpha k\cos[k\log(1+\alpha\varphi) - \gamma_1] + \left\{\alpha\left(-\frac{5}{2}+k\right) - \frac{1}{2}(1+\alpha\varphi)\cot\varphi\right\}\sin[k\log(1+\alpha\varphi) - \gamma_1]\} + \frac{2\alpha^2 k^2 C_2}{Ed^{1/2}(\sin\varphi)^{1/2}}(1+\alpha\varphi)^{-(7/2)-k}\{-\alpha k\cos[k\log(1+\alpha\varphi) - \gamma_2] + \left[\alpha\left(\frac{5}{2}+k\right) + \frac{1}{2}(1+\alpha\varphi)\cot\varphi\right]\sin[k\log(1+\alpha\varphi) - \gamma_2]\} \quad \ldots\ldots [5]$$

$$^{\circ}N_1 = -{}^{\circ}Q\cot\varphi \quad \ldots\ldots [6]$$

$$^{\circ}N_2 = -{}^{\circ}Q' \quad \ldots\ldots [7]$$

[4] The index $^{\circ}( )$ indicates the homogeneous solution.

[5] Certain minor terms have been omitted in these results.

[6] Differentiation with respect to φ is represented by the prime $( )'$.

$$^{\circ}M_1 = (D/a)(^{\circ}\chi' + \mu\,{}^{\circ}\chi\cot\varphi) \quad \ldots\ldots [8]$$

$$^{\circ}M_2 = (D/a)(^{\circ}\chi\cot\varphi + \mu\,{}^{\circ}\chi') \quad \ldots\ldots [9]$$

where C_1, C_2, γ_1, and γ_2, are constants of integration which must be determined from the loading and boundary conditions;

$$k = (1/\alpha)(a/d)^{1/2}[3(1-\mu^2)]^{1/4} \quad \ldots\ldots [10]$$

μ = Poisson's ratio

χ = rotation due to deformation of the meridional tangent, or the normal to the shell surface, taken positive clockwise

$$D = E\delta^3/[12(1-\mu^2)] \quad \ldots\ldots [11]$$

= modulus of flexural rigidity, a variable because of the presence of δ.

It may be observed from the foregoing formulas that the pole ($\varphi = 0$), is a singular point, but its local character does not affect the accuracy of the results at other points.

This solution applies to the general case of the shell of two boundaries. Constants C_1 and γ_1 are determined from conditions which prevail on the lower boundary, and C_2 and γ_2 from conditions on the upper boundary.

When the shell is closed, that is, extends all the way to the pole, $C_2 = \gamma_2 = 0$ and only the constants C_1 and γ_1 must be evaluated. A proof may be given for this based on consideration of the fact that for the closed shell the shearing force Q and the rotation of the meridional tangent χ must both be zero at the pole.

Particular Solution When the Load Is Distributed Over the Surface of the Shell

The particular integral for the nonhomogeneous differential equations will of course depend upon the type of loading to which the shell surface is subjected.

In general it is not possible to obtain a particular integral to the lengthy nonhomogeneous equations by means of any of the recognized methods, and resort must be made to approximations.

For this purpose for most types of loading it is satisfactory to use the membrane forces as the particular solutions for N_1 and N_2, and take Q as equal to zero.

Example

The foregoing solution will now be applied to a numerical example.

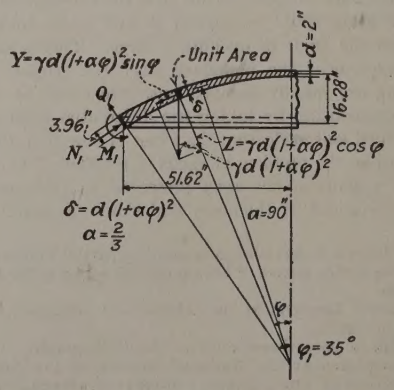

Fig. 2 Shell of Example Solved for Dead Load

Consider the closed shell of concrete with built-in boundary, as illustrated in Fig. 2, when the loading is taken as its own weight.

The numerical data for this shell are: $a = 90$ in. radius;

$\varphi_1 = 35$ deg, angle of opening; $d = 2$ in., thickness at pole; $\alpha = 2/3$; $E = 3{,}000{,}000$ psi; $\mu = 0.15$; $\gamma = 0.0868$ lb per cu in., weight; and from Equation [10], $k = 13.1676$.

Calculation of Constants of Integration

As shown in Fig. 2, the tangential and normal components of the surface load are

$$Y = \gamma d(1 + \alpha\varphi)^2 \sin\varphi \quad [12]$$

$$Z = \gamma d(1 + \alpha\varphi)^2 \cos\varphi \quad [13]$$

For a fixed-edge shell the conditions to be satisfied on the boundary are

$$v = 0, \quad w = 0, \quad \text{and} \quad \chi = 0 \quad [14][15][16]$$

As shown in the Appendix, the expression for w is

$$w = (a/E\delta)(N_2 - \mu N_1) - v\cot\varphi \quad [75]$$

This will be satisfied on the boundary where both v and w are zero if

$$N_2 - \mu N_1 = 0 \quad [17]$$

Since N_1 and N_2 denote the general solution for these forces this equation may be written

$$^{\circ}N_2 + {}^{DL}N_2 - \mu(^{\circ}N_1 + {}^{DL}N_1) = 0 \quad [18]$$

or

$$^{\circ}N_2 + {}^{m}N_2 - \mu(^{\circ}N_1 + {}^{m}N_1) = 0 \quad [19]$$

where the index $^{DL}(\ )$ indicates particular solution for dead load, and $^{m}(\ )$ indicates the membrane force.

Substituting the known expressions for the membrane forces for a sphere, and the values of the homogeneous solution from Equations [6] and [7] yields

$$-{}^{\circ}Q' + \frac{F(\varphi)}{\sin^2\varphi} - Za - \mu\left[-{}^{\circ}Q\cot\varphi - \frac{F(\varphi)}{\sin^2\varphi}\right] = 0 \quad [20]$$

where

$$F(\varphi) = a\int_0^{\varphi}(Y\sin\varphi + Z\cos\varphi)\sin\varphi d\varphi \quad [21]$$

For dead load as given by Equations [12] and [13] this becomes

$$F(\varphi) = a\gamma d[2\alpha(1 + \alpha\varphi)\sin\varphi - (1 + \alpha\varphi)^2\cos\varphi + 2\alpha^2\cos\varphi + 1 - 2\alpha^2] \quad [22]$$

For the calculation of the constants, it will be convenient to let

$$\gamma_1 = k\log(1 + \alpha\varphi_1) + \psi_1 \quad [23]$$

where φ_1 is the angle of opening of the shell and ψ_1 is another constant.

Substituting for $^{\circ}Q$ and $^{\circ}Q'$ in Equation [20] and performing the numerical computations for $\varphi = \varphi_1 = 35$ deg gives

$$C_1(\sin\psi_1 + 0.96510\cos\psi_1) = -0.00534 \quad [24]$$

Equation [16] may be written as

$$^{\circ}\chi + {}^{DL}\chi = 0 \quad [25]$$

In Equation [54] as derived in the Appendix the value of $^{DL}\chi$, if the particular solution for Q is neglected, is found to be

$$E\,{}^{DL}\chi = (1/\delta)\Phi(\varphi) \quad [26]$$

where $\Phi(\varphi)$ as given by Equation [55] becomes for dead load

$$\Phi(\varphi) = -(1 + \mu)\frac{\delta'}{\delta}\frac{F(\varphi)}{\sin^2\varphi} + a\gamma d(2 + \mu)(1 + \alpha\varphi)^2\sin\varphi \quad [27]$$

Substituting Equations [4] and [26] in [25] and performing the numerical work for $\varphi = \varphi_1 = 35$ deg yields

$$C_1\sin\psi_1 = 0.001054 \quad [28]$$

Solving Equations [24] and [28] gives

$$C_1 = -0.006707 \quad [29]$$

$$\psi_1 = -(9°2'32'') = -0.15782 \text{ radians} \quad [30]$$

from which

$$\gamma_1 = 4.34067 \quad [30a]$$

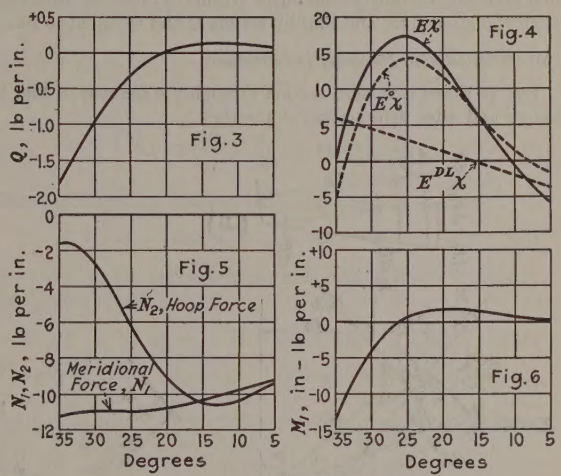

FIG. 3 SHEARING FORCE Q

FIG. 4 ROTATION OF MERIDIONAL TANGENT

FIG. 5 HOOP FORCE N_2

FIG. 6 MERIDIONAL BENDING MOMENT M_1

When the constants C_1 and γ_1 are determined, the solution as represented by Figs. 3, 4, 5, and 6 may be completed by substitution in the appropriate equations.

In computing the values of the bending moments it is necessary to have $^{DL}\chi'$. Much labor will be saved if, instead of differentiating Equation [26], the curve for $E\,{}^{DL}\chi$ as shown in Fig. 4 is first represented by an empirical polynomial of the second degree and then differentiated.

Appendix

(*a*) *Equilibrium Equations*

The equations of equilibrium of the element shown in Fig. 1 are

$$(Q\sin\varphi)' + N_1\sin\varphi + N_2\sin\varphi + Za\sin\varphi = 0 \quad [31]$$

$$(N_1\sin\varphi)' - N_2\cos\varphi - Q\sin\varphi + Ya\sin\varphi = 0 \quad [32]$$

$$(M_1\sin\varphi)' - M_2\cos\varphi - Qa\sin\varphi = 0 \quad [33]$$

Expressions for N_1 and N_2 may be obtained as follows: Multiply Equation [31] by $\cos\varphi$ and Equation [32] by $\sin\varphi$ and add.

$$(N_1\sin\varphi)'\sin\varphi + N_1\sin\varphi\cos\varphi + (Q\sin\varphi)'\cos\varphi - Q\sin^2\varphi + a(Y\sin\varphi + Z\cos\varphi)\sin\varphi = 0$$

Integrating this equation gives

$$(N_1 \sin \varphi + Q \cos \varphi)\sin \varphi + F(\varphi) = 0 \quad \ldots\ldots [34]$$

where $F(\varphi)$ is as given by Equation [21].

From Equations [34] and [31] is obtained

$$N_1 = -Q \cot \varphi - F(\varphi)/\sin^2 \varphi \quad \ldots\ldots [35]$$

$$N_2 = -Q' + [F(\varphi)/\sin^2 \varphi] - Za \quad \ldots\ldots [36]$$

Equations [35] and [36] become Equations [6] and [7], respectively, when taken in the homogeneous form. On the other hand if the shell is a membrane and incapable of resisting shear and moment, the membrane forces N_1 and N_2 may be obtained from Equations [35] and [36] by setting Q and Q' equal to zero.

(*b*) *Relationships Involving Deformations*

Fig. 7*a* shows a vertical section through the element of Fig. 1 before and after deformation of the shell.

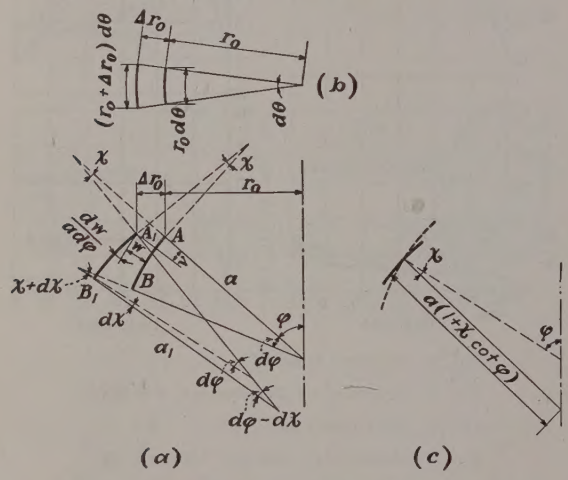

FIG. 7 DEFORMATION OF THE ELEMENT

[(*a*) Rotation of meridional tangent, also radius of curvature in direction of coordinate φ; (*b*) View of upper edge of element; (*c*) Radius of curvature in direction of coordinate θ.]

A point originally at A on the upper edge of the element moves to A_1 undergoing the two displacements v and w.

Point B on the lower edge of the shell moves to B_1 after receiving displacements of $v + v'd\varphi$ and $w + w'd\varphi$.

The tangent to the meridian drawn at A has thus rotated counterclockwise through an angle v/a due to v, and clockwise through an angle w'/a because of the variation in w. The total rotation is thus

$$\chi = (w' - v)/a \quad \ldots\ldots [37]$$

It will be convenient to use χ as one of the unknowns rather than v and w separately.

Consider the curvature of the normal section of the surface whose plane contains the tangent to the meridian curve.

Normals drawn from the points B and B_1, Fig. 7*a*, will intersect at an angle $\chi + d\chi$. Normals from the points A_1 and B_1 will then intersect at an angle $d\varphi - d\chi$. The length of the arc AB then is

$$ad\varphi = a_1 (d\varphi - d\chi)$$

if a_1 be the radius of curvature after deformation, and if any change in length between AB and A_1B_1 be neglected.

The curvature for the deformed element then is

$$1/a_1 = (1/a)(1 - \chi')$$

and the change of curvature in the φ-direction is

$$\kappa_1 = \chi'/a = (w' - v)'/a^2 \quad \ldots\ldots [38]$$

Now consider the curvature of the section at right angles to φ-direction. Neglect the slight displacement of the element and regard only the effect of the rotation. The new radius of curvature extends from the element to the axis of the sphere, and as shown in Fig. 7*c*, has a length approximately equal to $a(1 + \chi \cot \varphi)$.

The change of curvature in the θ-direction is then found to be approximately

$$\kappa_2 = (\chi \cot \varphi)/a = [(w' - v)/a^2]\cot \varphi \quad \ldots\ldots [39]$$

Thus for the bending moments

$$M_1 = D(\kappa_1 + \mu\kappa_2) = (D/a)(\chi' + \mu\chi \cot \varphi)$$
$$= (D/a^2)[(w' - v)' + \mu(w' - v)\cot \varphi] \quad \ldots\ldots [40]$$

$$M_2 = D(\kappa_2 + \mu\kappa_1) = (D/a)(\chi \cot \varphi + \mu\chi')$$
$$= (D/a^2)[(w' - v)\cot \varphi + \mu(w' - v)'] \quad \ldots\ldots [41]$$

As shown in Fig. 7*a* the length of the arc A_1B_1 is given approximately by $(ad\varphi + v'd\varphi)[(a + w)/a]$. Neglecting terms of higher order the increase in the length of the element then is $(v' + w)d\varphi$ and the deformation is

$$\epsilon_1 = (v' + w)/a \quad \ldots\ldots [42]$$

In the θ-direction the new length of arc as shown in Fig. 7*b* is $(r_0 + \Delta r_0)d\theta$. The increase in length then is

$$\Delta r_0 d\theta = (v \cos \varphi + w \sin \varphi)d\theta$$

and the elongation is

$$\epsilon_2 = (1/a)(v \cot \varphi + w) \quad \ldots\ldots [43]$$

The edge forces are thus seen to be

$$N_1 = [E\delta/(1 - \mu^2)](\epsilon_1 + \mu\epsilon_2)$$
$$= [E\delta/a(1 - \mu^2)][v' + w + \mu(v \cot \varphi + w)] \quad \ldots\ldots [44]$$

$$N_2 = [E\delta/(1 - \mu^2)](\epsilon_2 + \mu\epsilon_1)$$
$$= [E\delta/a(1 - \mu^2)][v \cot \varphi + w + \mu(v' + w)] \quad \ldots\ldots [45]$$

The seven Equations [31], [32], [33], [40], [41], [44], and [45] make it possible to solve for the seven aforementioned unknowns contained therein. These equations will first be reduced to two equations in the unknowns Q and χ.

(*c*) *Differential Equations of the Shell*

Substituting Equations [40] and [41] into Equation [33] gives the first differential equation of the shell.

$$\chi'' + \chi'[\cot \varphi + 3(\delta'/\delta)] - \chi[\cot^2 \varphi + \mu - 3\mu(\delta'/\delta)\cot \varphi] = a^2Q/D \quad \ldots\ldots [46]$$

For the second equation proceed as follows. Combine Equations [35] and [36] as here indicated

$$N_2 - \mu N_1 = -Q' + \mu Q \cot \varphi + (1 + \mu)\frac{F(\varphi)}{\sin^2 \varphi} - Za \quad \ldots [47]$$

$$N_2' - \mu N_1' = -Q'' + \mu Q' \cot \varphi - \mu Q \csc^2 \varphi + \frac{1 + \mu}{\sin^2 \varphi} F'(\varphi) - \frac{2(1 + \mu)\cos \varphi F(\varphi)}{\sin^3 \varphi} - Z'a \quad \ldots [48]$$

$$N_1 - N_2 = Q' - Q \cot \varphi - \frac{2F(\varphi)}{\sin^2 \varphi} + Za \quad \ldots\ldots [49]$$

Combine Equations [44] and [45]

$$v' + w = (a/E\delta)(N_1 - \mu N_2) \quad [50]$$

$$v \cot \varphi + w = (a/E\delta)(N_2 - \mu N_1) \quad [51]$$

Subtract Equation [51] from [50]

$$v' - v \cot \varphi = (a/E\delta)(1 + \mu)(N_1 - N_2) \quad [52]$$

Differentiating Equation [51] and subtracting the product of Equation [52] times cot φ gives

$$w' - v = (a/E\delta)[-(\delta'/\delta)(N_2 - \mu N_1) + (N_2' - \mu N_1') - (1 + \mu)(N_1 - N_2)\cot \varphi] \quad [53]$$

Substituting Equations [47], [48], [49], and [37] into Equation [53] gives the second differential equation of the shell.

$$Q'' + Q'[\cot \varphi - (\delta'/\delta)] - Q[\cot^2 \varphi - \mu - \mu(\delta'/\delta)\cot \varphi] = -E\delta\chi - (1 + \mu)\frac{\delta'}{\delta}\frac{F(\varphi)}{\sin^2 \varphi} + a[(1 + \mu)Y + \frac{\delta'}{\delta}Z - Z'] = -E\delta\chi + \Phi(\varphi) \quad [54]$$

where

$$\Phi(\varphi) = -(1 + \mu)\frac{\delta'}{\delta}\frac{F(\varphi)}{\sin^2 \varphi} + a[(1 + \mu)Y + \frac{\delta'}{\delta}Z - Z'] \quad [55]$$

Equations [46] and [54] are general, and will apply to any type of variation in δ, and to any type of external load.

(d) Solution of the Homogeneous Equations

Equation [46] is homogeneous in the variables χ and Q, and Equation [54] may be made thus by omitting the term $\Phi(\varphi)$ which involves the components Y and Z of the external load.

An approximate solution may be obtained to Equations [46] and [54] based on the assumption that the expressions which represent χ and Q are of such a type that the first two terms on the left of Equations [46] and [54] are of much more importance than the third term, and that this third term may therefore be neglected.

Equation [46] becomes then when the term in χ is omitted

$$\chi'' + \chi'[\cot \varphi + 3(\delta'/\delta)] = a^2Q/D \quad [56]$$

The term in χ' may be removed from this equation by the following substitution giving an equation in the new variable $\bar{\chi}$. Let

$$\chi = \bar{\chi}e^{-1/2\int(\cot\varphi + 3\delta'/\delta)d\varphi} = \bar{\chi}/(\delta^3 \sin \varphi)^{1/2} \quad [57]$$

Substitute in Equation [56] and neglect terms in the variable itself.

$$\bar{\chi}''/(\delta^3 \sin \varphi)^{1/2} = a^2Q/D \quad [58]$$

Omitting the term in Q, Equation [54] becomes

$$Q'' + Q'[\cot \varphi - (\delta'/\delta)] = -E\delta\chi \quad [59]$$

The term in Q' is removed by the substitution

$$Q = \bar{Q}e^{-1/2\int(\cot\varphi - \delta'/\delta)d\varphi} = \bar{Q}(\delta/\sin \varphi)^{1/2} \quad [60]$$

Substituting in Equation [59] and neglecting terms in the variable itself gives

$$\bar{Q}''(\delta/\sin \varphi)^{1/2} = -E\delta\chi \quad [61]$$

Equation [60] should now be substituted into [58] and Equation [57] into [61].

$$\bar{\chi}''\delta = [12a^2(1 - \mu^2)/E]\bar{Q} \quad [62]$$

$$\bar{Q}''\delta = -E\bar{\chi} \quad [63]$$

Eliminating $\bar{\chi}$ between Equations [62] and [63] gives

$$\delta^2\bar{Q}^{IV} + 2\delta\delta'\bar{Q}''' + \delta\delta''\bar{Q}'' + 12a^2(1 - \mu^2)\bar{Q} = 0 \quad [64]$$

Substituting the value of δ as given by Equation [1] gives

$$(1 + \alpha\varphi)^4\bar{Q}^{IV} + 4\alpha(1 + \alpha\varphi)^3\bar{Q}''' + 2\alpha^2(1 + \alpha\varphi)^2\bar{Q}'' + 12\,(a^2/d^2)(1 - \mu^2)\bar{Q} = 0 \quad [65]$$

This equation is reduced to one with constant coefficients by the following substitutions. Let

$$u = 1 + \alpha\varphi \quad [66]$$

$$u^4\frac{d^4\bar{Q}}{du^4} + 4u^3\frac{d^3\bar{Q}}{du^3} + 2u^2\frac{d^2\bar{Q}}{du^2} + 4k^4\bar{Q} = 0 \quad [67]$$

where k is given by Equation [10]. Now let

$$u = e^t \quad [68]$$

$$\frac{d^4\bar{Q}}{dt^4} - 2\frac{d^3\bar{Q}}{dt^3} + \frac{d^2\bar{Q}}{dt^2} + 4k^4\bar{Q} = 0 \quad [69]$$

The corresponding characteristic equation has roots of

$$m = (1/2)[1 \pm (1 \pm 8ik^2)^{1/2}] \quad [70]$$

but since $1 \ll 8k^2$ the term unity under the radical will be neglected giving values of m of

$$m = (1/2) \pm (1 \pm i)k \quad [71]$$

The homogeneous value of the shearing force is thus found to be

$$\begin{aligned} {}^{\circ}Q &= \frac{d^{1/2}(1 + \alpha\varphi)}{(\sin \varphi)^{1/2}}\{A_1e^{[1/2+(1+i)k]t} + A_2e^{[1/2+(1-i)k]t} + A_3e^{[1/2-(1-i)k]t} + A_4e^{[1/2-(1+i)k]t}\} \\ &= \frac{d^{1/2}(1 + \alpha\varphi)}{(\sin \varphi)^{1/2}}\{e^{(1/2+k)t}(A_5 \cos kt + A_6 \sin kt) + e^{(1/2-k)t}(A_7 \cos kt + A_8 \sin kt)\} \\ &= \frac{d^{1/2}(1 + \alpha\varphi)}{(\sin \varphi)^{1/2}}[C_1e^{(1/2+k)t}\cos(kt - \gamma_1) + C_2e^{(1/2-k)t}\cos(kt - \gamma_2)] \end{aligned} \quad [72]$$

Upon substitution of $t = \log(1 + \alpha\varphi)$ the value for the shearing force as given by Equation [2] is obtained. The rotation of the meridional tangent χ may be obtained by substitution into Equations [60] and [61].

Combining Equations [42] and [43] gives

$$v' - v \cot \varphi = a(\epsilon_1 - \epsilon_2) \quad [73]$$

The integral of this is

$$v = C \sin \varphi + a \sin \varphi \int \frac{\epsilon_1 - \epsilon_2}{\sin \varphi}\,d\varphi \quad [74]$$

where C is the constant of integration.

Substituting Equations [44] and [45] into Equation [43] yields

$$w = (a/E\delta)(N_2 - \mu N_1) - v \cot \varphi \quad [75]$$

Summary

An approximate method of treating spherical shells of variable thickness, similar to that developed for spherical shells of constant thickness (10), is developed in this paper.

It has been shown by Hetényi (10) that the method of approximation as used here for shells of variable thickness will, when applied to the case of constant wall thickness, give results which

are almost identical with those given by the exact solution of the differential equations. Very likely this same relative accuracy will hold when the wall thickness varies. In all cases substitution in the differential equations constitutes the final criterion.

BIBLIOGRAPHY

1 "Mathematical Theory of Elasticity," by A. E. H. Love, University Press, Cambridge, England, 4th edition, 1927, p. 535.

2 "Spannungen in Kugelschalen," by H. Reissner, Müller-Breslau-Festschrift, A. Kröner, Leipzig, Germany, 1912, p. 181.

3 "Über die Asymptotische Integration, Etc.," by O. Blumenthal *Zeitschrift für Mathematik und Physik*, vol. 62, 1914, p. 343.

4 "Das Elastizitätsproblem für dünne Schalen," by E. Meissner. *Physikalische Zeitschrift*, vol. 14, 1913, p. 343.

5 "Zur Festigkeitsberechnung von Hochdruckkesseltrommeln," by E. Meissner, *Schweizerische Bauzeitung*, vol. 86, July, 1925, p. 1.

6 "Über die Festigkeit Achsensymmetrischer Schalen," by J Geckeler, *Forschungsarbeiten auf dem Gebiete des Ingenieurwesens*, Heft 276, 1926.

7 "Über Elastizität und Festigkeit Dünner Schalen," by E. Meissner, Veirteljahrschrift der Naturforschende Gesellschaft. Zurich, vol. 60, 1915, p. 23.

8 "Statik und Dynamik der Schalen," by W. Flügge, Julius Springer, Berlin, Germany, 1934.

9 "Berechnung von Behältern," by T. Pöschl, Julius Springer Berlin, Germany, 1926.

10 "Spherical Shells Subjected to Axial Symmetrical Bending," by M. Hetényi, Publication of International Ass'n for Bridge and Structural Engineering, Eidgen. Techn. Hochschule, Zurich. Switzerland, vol. 5, 1938.